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ADVANCE STUDY OF HURWITZ THEOREM

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ABSTRACT

Diophantine's approximation incorporates Hurwitz's theorem- established in 1891 that focuses more on the irrationality of numbers in line with rationality. The proposition of "approximation of irrational numbers by rational numbers"(https://math.stackexchange.com) is possible because the rational numbers are condensed on real line. The prominent part of Diophantine approximations is the advanced study of Hurwitz theorem that is irrationals by rationales. In this sense, the paper aims to equate, the distance between a given real numbers ξ and a rational number h/k, and the denominator k of the approximate. The paper is significant for upbringing the concept of approximation which is enhanced as with the increasing size of denominator. The study, hence, applies Hurwitz's theoretical underpinnings to go through Diophantine's approximation with its proper application in dealing with, for discussion.

KEYWORDS: Approximation, Hurwitz Theorem, Irrational and Rational Number & Continued Fraction

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Original Article

1. INTRODUCTION

The concept of Diophantine approximation was introduced by Greek Mathematician Diophantus of Alexandria in 250 A.D. The Greek Mathematician Diophantus originated the concept "fractions as numbers". "Arithmatika" has been the significant works of Diophantine employing its major task on "Algebra in Greek mathematics"(Wikipedia). Diophantine approximation is the theory of approximation of irrational numbers by rational numbers. Almost all the work of Diophantus are in the verge of extinction. Even the testimonies of lemmas are becoming problematic that it would be very much difficult to easily believe on Diophantus work. "Lagrange's theorem", the proposition of Lagrange becomes quite apt here to sensitize the lost one theorems of Diophantaus issuing on "that every integer is the sum of four squares." "Given any positive rationals a, b with a > b, there exist positive rationals c, d such that a^3-b^3=c^3+d^3". After that "lemma" was examined by Vieta and Fermat who ultimately resolved, in the late nineteenth century in a difficult way. It appears unlikely in the sense that Diophantus essentially comprised proofs related to "lemmas". Then Diophantus tried to practicalize "complex algebraic problems" as a part of his analysis with the spirit of dealing for searching integer solution. His equations can be inferred that they possess "polynomial equations with integer coefficients to which only integer solutions are sought" (web). One of the seen attributes of his equations is having restricted variables to "integral values". For Instance: $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$.

Diophantus asked: integer solutions could there be for n=4, for the aforementioned n what could be the distinct values. For clarifying with a proper example, it may consist of three dissimilar values as: $\frac{1}{5}+\frac{1}{20}=\frac{1}{4},\frac{1}{6}+\frac{1}{12}=\frac{1}{4},\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$. Another, famous Diophantine equation is $x^n + y^n = z^n$ which was the subject of Fermat's Last Theorem. In 1570, A. D. Arthmatika was first rendered from Greek in to Latin by Bombelli, but it remained

unpublished. In 1575, Xylander published a book named "Editio Princeps of Arithmetika". In 1621, Bachet translated one of his best books called Arthmatika in Latin script. Similarly, in 1637, another mathematician called Pierre de Fermat claims that his theory needs to be testified. In 1695, John Wallis laid some of the basic work for continued fractions in his book "Opera Mathematica". In 1761, Johann Heinrich Lambert proved that "π is irrational. The necessity of approximating irrational number arises in many contexts in mathematics. The Pythagorean School used to believe that the only number existed is the now called rational numbers, until someone" (web). Thus, the early Greeks then faced the equation x^2-$2y^2$=1 for the perception $\sqrt{2}$ by discovering a way to build an "infinite sequence" of rationales that estimates $\sqrt{2}$ properly with each term. This is discussed in background analysis so as to show as the rational field that Q does not content the "least upper bound property" (web). Pell's Equation that concerns with x^2-ny^2=1 also becomes inspiration for "Diophantine approximation" especially for getting x and y as integer values. What people have realized that $\sqrt{n}$ arbitrarily remained closed about and during 600 CE. Though, the presence of such key was not confirmed till his proof published in 1768 as lagrange. The study tries to present a solid and clear evidence— a weighty result— that relates with Dirichlet's approximation theorem. The approximation of a real number by a rational is a problem that arose in arithmetic with the discovery "of irrational numbers, such as the square root of n, when n is a positive integer that is not an exact square". This finding has given birth to the field of rational is called Diophantine approximation of such numbers. It is the procedure of getting ways out sequentially proper rational approximations to irrational numbers(math.columbia.edu).

The primary aim of the theory of Diophantine approximation is to compare, the distance between a given real numbers ξ and a rational number $\frac{h}{k}$ with denominator k of the approximate. Here, an approximation is measured as sharp if $\left|\xi - \frac{h}{k}\right|$ is small paralleled to k. Example: Suppose we are trying to make a rectangle of a circle that is finding a rectangle whose area is equal to the area of a given circle. The ratio of the perimeter of the rectangle to the diameter of the original circle will be an irrational number. So, an approximation will have to be used in computations. Dirichlet developed an approximation theorem in 1840A.D. becoming a starting step for Diophantine approximation. The subsequent theorems belong to Diophantine approximation.

2. THE IRRATIONALS AND DIRICHLET'S THEOREM

Theorem 1 (Dirichlet's Approximation Theorem: 1830 A.D)

If n refers a non-negative "integer and ξ is a real number, there is a rational number $\frac{h}{k}$ such that $0 < k \leq n$ and $\left|\xi - \frac{h}{k}\right| \leq \frac{1}{k(n+1)}$ "

Proof: Consider a n^{th} order farey sequence. "Let $\frac{h}{k}$ and $\frac{c}{d}$ with $\frac{h}{k} < \frac{c}{d}$ be two consecutive fractions in that sequence such that the closed interval $\left[\frac{h}{k}, \frac{h+c}{k+d}\right]$ containing ξ then $\left|\xi - \frac{h}{k}\right| \leq \left|\frac{h}{k} - \frac{h+c}{k+d}\right| = \left|\frac{hk+hd-hk-kc}{k(k+d)}\right| = \left|\frac{hd-kc}{k(k+d)}\right| = \left|\frac{1}{k(k+d)}\right| \leq \left|\frac{1}{k(n+1)}\right|$ " (Ivan,2015). Here, Dirichlet's Approximation Theorem becomes subsequent to corollary 1.

Corollary 1: (Dirichlet's Theorem, 1842 A.D)

If ξ is real and irrational, it sets up unlimited separate "rational numbers $\frac{h}{k}$ such that $\left|\xi - \frac{h}{k}\right| < \frac{1}{k^2}$."

Proof: suppose there are only finitely many rationals $\frac{h_1}{k_1}, \frac{h_2}{k_2}, \ldots, \frac{h_j}{k_j}$, satisfying $\left|\xi - \frac{h}{k}\right| < \frac{1}{k_i^2}$ for $1 \leq i \leq j$,

consequently, since ξ is irrational, there exist a positive integer n such that the inequality $\left|\xi-\frac{h_1}{k_1}\right| \leq \frac{1}{n+1}$ holds, for $1 \leq i \leq j$. However, this contradicts Dirichlet's theorem, which asserts that, for this n, "there exists a rational number $\frac{h}{k}$ with $k \leq n$ such that $\left|\xi-\frac{h_1}{k_1}\right| \leq \frac{1}{n+1} < \frac{1}{k^2}$" (Casssel, 1957). To be more specific, Dirichlet's corollary 1 contradicts the theorem.

Theorem 2: A real numbers ξ becomes irrational when there happens a situation to that of similar to "infinitely many rational numbers $\frac{h}{k}$ such that $\left|\xi-\frac{h}{k}\right| \leq \frac{1}{k^2}$".

According to Cassel J.W.S. in his book *An Introduction to Diophantine Approximation* "One direction in this statement follows directly from the theorem we just proved. A complete proof can be found in" (1957). Hence, the theorem is remarkable to write because of its proper application. It happens that rational numbers are distinct from irrational numbers because of having approximation with infinitely an excessive portion of "rational in It appears that irrational numbers can be distinguished from rational numbers by the fact that they can be approximated by infinitely many rational numbers $\frac{h}{k}$ with n error less than $\frac{1}{k^2}$ " (Cassel, 1957).

3. HURWITZ'S THEOREM AND CONTINUED FRACTIONS

Definition: Let the finite continued fraction expansion for a real number ξ =

$$u_0+\cfrac{1}{u_1+\cfrac{1}{u_2+\cfrac{1}{u_{n-1}+\cfrac{1}{u_n}}}} \ddots$$

is denoted by $\xi = [u_0; u_1, u_2, u_3, \ldots, u_n]$ where $u_i s$ are integer parts. Clearly, $u_i \epsilon Z$ for all i and $u_i > 0$ for all $i \geq 1$. If the term of continued fraction goes infinite in number then it is called simple infinite continued fraction of real number. It is denoted by $\xi = [u_0; u_1, u_2, u_3, \ldots]$.

Definition: The number $\frac{h_n}{k_n} = [u_0; u, u_2, u_3, \ldots, u_n] = \frac{u_n h_{n-1}+h_{n-2}}{u_n k_{n-1}+k_{n-2}}$ is called the n^{th} convergent of ξ, where $h_{-2} = 0, h_{-1} = 1, k = 1, k = 0, gives\ h_0 = u_0, k_0 = 1$.

The integer u_n in the continued fraction expansion is called n^{th} partial quotient and $\xi_n = u_n, u_{n+1}, \ldots.]$, is called the n^{th} total quotient.

As with the aforementioned definition, it can be proved that "an array" of useful "and interesting properties of continued fractions, all of which will eventually lead us to the development" of Hurwitz's "theorem".

Lemma : for $n \geq 0$

$p_n = a_n p_{n-1} + P_{n-2}, q_n = q_n\, q_{n-1} + q_{n-2}$ (Cassel, 1957)

Corollary 2 : for $n \geq -1, q_n p_{n-1} - p_n q_{n-1} = (-1)^n$(Rockett And Szusz,1992)

Corollary 3: For n ≥ 0, $q_n p_{n-2} - p_n q_{n-2} = (-1)^{n-1} a_n$ (Rockett And Szusz,1992)

Corollary 4: The convergent $\frac{p_k}{q_k}$ satisfies the following inequalities:

1. $\frac{p_o}{q_o} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots$

2. $\frac{p_1}{q_1} < \frac{p_3}{q_3} < \frac{p_5}{q_{54}} > \ldots$

3. If we supply any non-negative even number 'n' and odd number 'm', we can find: $\frac{p_n}{q_n} < \frac{p_m}{q_m}$

Proof: The division of two sides $q_{(n-2)}q_n$ in above corollary 3, provides the result as $\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1}a^n}{q_{n-2}q_n}$.

It indicates that if $n \geq 2$ and n is even number, we get *(n-2)*th term that becomes less than n^{th} term. In the same way, if $n \geq 3$ and n is odd number. We find that $(n-2)^{th}$ term becomes larger than n^{th} term. Such theoretical condition creates 1 and 2 parts. It is notable here that, part 3 of corollary 4 is true for " arbitrary pair of odd and even numbers" (Cassel, 1957) which we can generally indicate $\frac{p_{m-1}}{q_{m-1}} < \frac{p_m}{q_m}$ becomes true for total odd number 'm' combining with 1 and 2 part to generate part 3. In this way, merely must represent, $\frac{p_{m-1}}{q_{m-1}} < \frac{p_m}{q_m}$. This implies "$q_m p_{m-1} - p_m q_{m-1} = (-1)^m < 0$" (Ivan, 2015). The relation here reproduces 3 part of this theorem.

Corollary 5

Let a_o be an integer and $a_1, a_2, a_3, \ldots$ be positive integers, define $\frac{p_n}{q_n}$ that becomes convergent of the continued fractions well-defined by $\{a_n\}_{n=0}^{\infty}$.And, $lim_{n\to\infty} \frac{p_n}{q_n}$ occurs with its irrational value. Contrarily, for ξ irrational, there happens a distinctive integer a_0 and distinctive positive integers $a_1, a_2, \ldots$ such that $\xi = lim_{n\to\infty} \frac{p_n}{q_n}$.

Proof: By the results, we have proved:

$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \frac{p_1}{q_1}$, it is clear that both limits: $lim_{neven\to\infty} \frac{p_n}{q_n}$ and $lim_{nodd\to\infty} \frac{p_n}{q_n}$, exists. Then furthermore, their recurrence relations make both limits equal. I place $\xi = lim_{n\to\infty} \frac{p_n}{q_n}$ and compute: $\left|\xi - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$.

Since p_n, q_n are comparatively prime number, there occurs infinite rational numbers $\frac{h}{k}$ as such $\left|\xi - \frac{h}{k}\right| < \frac{1}{k^2}$. So ξ in this case becomes irrational number. In the opposite way, suppose ξ is irrational number, $a_0 = \lfloor \xi \rfloor$, suppose $\xi_1 = a_0 + \frac{1}{\xi_1}$. We notice that $\xi_1 > 1$ is irrational. For $k \geq 1$. Let $a_k = \lfloor \xi_k \rfloor$ and $\xi_k = a_k + \frac{1}{\xi_{k+1}}$. It can be perceived that $a_k \geq 1$. $\xi_{k+1} > 1$ and ξ_{k+1} becomes irrational. What I want to intend here is: $\xi = [a_0; a_1, a_2, a_3, \ldots]$ using the recurrence we have proven before with $\xi = [a_0; a_1, a_2, a_3, \ldots]$ we find: Using the recurrence we have proven before with $\xi = [a_0; a_1, a_2, a_3, \ldots]$ we find:

$$q_n\xi - p_n = q_n . \frac{(\xi_{n+1}p_n + p_{n-1})}{(\xi_{n+1}q_n + q_{n-1})} - p_n$$

$$= \frac{q_n(\xi_{n+1}p_n + p_{n-1}) - p_n(\xi_{n+1}q_n + q_{n-1})}{\xi_{n+1}q_n + q_{n-1}}$$

$$= \frac{(-1)^n}{\xi_{n+1}q_n + q_{n-1}}$$

Hence $\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$ Which infers $lim_{n\to\infty} \frac{p_n}{q_n} = \xi$. Ultimately, it is left out to prove the integers $a_0, a_1 \geq 1, a_2 \geq 2, \ldots$ that are quite perfectly measured. In perspective of $\xi = [a_0, a_1, a_2, \ldots] = \left|a_0 + \frac{1}{[a_1, a_2, \ldots]}\right|$ and $0 \leq \xi - a_0 < 1$, we find

$a_0 = \lfloor\xi\rfloor$ which implies that a_0 is unique and $\xi_1 = [a_1, a_2, \dots]$ is uniquely determined by ξ. Because $a_1 = \lfloor\xi_1\rfloor$, a_1 becomes unique. The expression as mentioned here validates the corollary. Henceforth, the aforesaid theorems and corollaries show proper tools for developing Hurwitz's Theorem. Properly remembering and analyzing the part 2 that we have obtained a bond $\frac{1}{q^2}$ for numbers focuses on this $\left|\xi - \frac{p}{q}\right|$ irrationalξ, where Vahlen and Borel,—the great mathematician of the time— successively found the bond in the theorem of Dirichlet. Furthermore, this becomes tough applying the technique of continued fractions. Yet the above said "improvements of the bonds culminated with Hurwitz's theorem, showing that this Dirichlet-type inequality's bond cannot be improved any further. Here, we first present to improvements of the bond in order". (Rockett And Szusz,1992)

Theorem 3: (Hurwitz, 1891) Let us consider that ξ is an irrational, and:

- There are many infinite rational numbers $\frac{h}{k}$ as such: $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$
- If $root$ 5 replaces by $C > \sqrt{5}$, and irrational ξ occurs for that statement one does not exist.

Theorem 4

The theorem 4 was propounded by the mathematician, Vahlen in 1895. Let us consider " be an irrational number and denote two consecutive convergent of ξ as $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$. Then, at least one of them satisfies the inequality $\left|\xi - \frac{h}{k}\right| < \frac{1}{2k^2}$." (Schmidt, 1980).

Proof: The numbers $\xi - \frac{p_n}{q_n}, \xi - \frac{p_{n-1}}{q_{n-1}}$ are of opposite sign, hence we get

$\left|\xi - \frac{p_n}{q_n}\right| + \left|\xi - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n q_{n-1}} < \frac{1}{2q_n^2} + \frac{1}{2q_{n-1}^2}$.

Since $pq < \frac{(p^2+q^2)}{2}$if $p \neq q$ if $h \neq k$. Hence, either $\left|\xi - \frac{p_n}{q_n}\right| = \frac{1}{2q_n^2}$ or $\left|\xi - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{2q_{n-1}^2}$.

Theorem 5

Theorem 5 was developed by Borel, another mathematician of the time. Suppose, ξ is the number irrational that shows 3 successive convergent form of ξ "as $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$. Then, at least one of them satisfies the inequality $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$." (Schmidt, 1980).

Proof: put $\xi = [a_0; a_1, a_2, a_3, \dots]$, $\xi_i = [a_i; a_{i+1}, \dots]$ and $\beta_i = \frac{q_{i-2}}{q_{i-1}}$ where $\boldsymbol{i}$ *becomes* $n-1, n, n+1$

We have $\xi = [a_0; a_1, a_2, a_3, \dots a_n, \xi_{n+1}], qn\xi - p_n = \frac{(-1)^2}{\xi_{n+1}q_n + q_{n-1}}$

By lemma, thus $\left|\xi - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\xi_{n+1}q_n + q_{n-1})} = \frac{1}{q_n^2(\xi_{n+1} + \beta_{n+1})}$.

For completing the evidence, it becomes appropriate to prove that there cannot be three integers $n-1, n, n+1$ with $\beta_i + \xi_i < \sqrt{5}$ (1)

Suppose that, equation (1) were true for $i = n-1, n.$ Now $\xi_{n-1} = a_{n-1} + \frac{1}{\xi_n}$ and $\frac{1}{\beta_n} = \frac{q_{n-1}}{q_{n-2}} = a_{n-1} + \frac{q_{n-1}}{q_{n-2}} = a_{n-1} + \beta_{n-1}$, so $\frac{1}{\xi_n} + \frac{1}{\beta_n} = \beta_{n-1} + \xi_{n-1} \leq \sqrt{5}$.

Therefore, $1 = \xi_n.\frac{1}{\xi_n} \leq (\sqrt{5} - \beta_n)\left(\sqrt{5} - \frac{1}{\beta_n}\right)$, which is equivalent to $\beta_n^2 - \sqrt{5}\beta_n + 1 \leq 0$.

It follows that $\beta_n \geq \frac{\sqrt{5}-1}{2}$, since β_n is rational, $\beta_n > \frac{\sqrt{5}-1}{2}$. If equation (1) also were true for $i = n, n+1$, then $\beta_{n+1} > \frac{\sqrt{5}-1}{2}$. And therefore, $1 \leq a_n = \frac{q_n}{q_{n-1}} - \frac{q_{n-2}}{q_{n-1}}$

$= \frac{1}{\beta_{n+1}} - \beta_n < \frac{2}{\sqrt{5}-1} - \frac{\sqrt{5}-1}{2} < 1$, is an ambiguous. This affirms such proof of the theorem.

Theorem 6 (Legendre)

Let h, k be integers such that $k \geq 1$, and $\left|\xi - \frac{h}{k}\right| < \frac{1}{2k^2}$. Then $\frac{h}{k}$ is a convergent of ξ.

Theorem 7

Suppose that the continued fraction development for ξ is maintained by $\xi = [a_0; a_1, \dots a_N, 1,1, \dots]$, then: $lim_{n\to\infty} k_n^2 \left|\xi - \frac{h}{k}\right| = \frac{1}{\sqrt{5}}$

Theorem 8: (Hurwitz)

Let us consider that ξ is an irrational, and:

- There are many infinite rational numbers $\frac{h}{k}$ as such: $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$
- If $root$ 5 replaces by $C > \sqrt{5}$, and irrational ξ occurs for that statement one does not exist

Proof: Statement (1) subsequently adopts Borel's Theorem. Accordingly, Legendre's Theorem matches statements (2) and (7). And, if ξ becomes irrational with the form of $\xi = [a_0; a_1, \dots a_N, 1,1, \dots]$. Then by Legendre's Theorem above, all solutions of $\left|\xi - \frac{h}{k}\right| < \frac{1}{Ck^2}$ with $C\ is\ greater\ than\ root$ 5 that converges ξ, although as with 8 theorem, such inequality is accepted to finite convergent to ξ.

4. EXTENSION OF HURWITZ'S THEOREM

Theorem 9 (Hurwitz Theorem, 1906)

Theorem 9 is one of the prominent theorems of Hurwitz which was commonly developed by 1906. It emphasizes that for all irrational number ξ the inequality $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$ has infinitely many solutions. If ξ is equivalent with $\frac{\sqrt{5}-1}{2}$ i.e. a root of the quadratic equation $\xi^2 + \xi - 1 = 0$ the constant $\frac{1}{\sqrt{5}}$ cannot be improved. For the irrational $\xi = \frac{\sqrt{5}+1}{2} = [1; 1,1 \dots]$. The inequality $\left|\xi - \frac{h}{k}\right| < \frac{1}{Ck_n^2}$, is satisfied for at most finitely many reduced $\frac{h}{k}$ if and only if $C > \sqrt{5}$. If not, there are infinitely many solutions of $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{8}k^2}$, where the constant $\frac{1}{\sqrt{8}}$ cannot be improved. If ξ is equivalent to a root of $\varepsilon^2 + 2\xi - 1 = 0$. Otherwise, there are infinitely many solutions of $\left|\xi - \frac{h}{k}\right| < \frac{5}{\sqrt{221}k^2}$, where the constant $\frac{5}{\sqrt{221}}$ cannot be improved for ξ equivalent to root of $5\xi^2 + 11\xi - 5 = 0$ otherwise, there are infinitely many solutions of $\left|\xi - \frac{h}{k}\right| < \frac{13}{\sqrt{1517}k^2}$, where the constant $\frac{13}{\sqrt{1517}}$ cannot be improved for ξ equivalent to a root of $13\xi^2 + 29\xi - 13 = 0$, so on indefinitely. The sequence of number $\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \frac{5}{\sqrt{221}}, \frac{5}{\sqrt{1517}}, \dots$ tend to $\frac{1}{3}$. (Havil,2012)

Theorem 10 (A.V. Prasad, 1948)

A.V. Prasad, the mathematician, further extended the theory of Hurwitz issuing on "any irrational ξ "becoming" least *m* pairs of relatively prime integers" *h*, *k* with *k*>*0* satisfying,$\left|\xi - \frac{h}{k}\right| \leq \frac{1}{C_m k^2}$, where $C_m = \frac{\sqrt{5}+1}{2} + \frac{a_{2m-1}}{b_{2m-1}}$, where $\frac{a_j}{b_j}$ is the j^{th} convergent to $\frac{\sqrt{5}-1}{2}$. Further if $\xi = \frac{\sqrt{5}-1}{2}$, then there are exactly *m* solutions. (A.V. Prasad, 1948). However, even A.V. Prasad's theorem had to modify in the case of constant$\xi = \frac{\sqrt{5}-1}{2}$. Hence, the curiosity regarding $\frac{\sqrt{5}-1}{2}$ has become the prime concern, raising a question as what may be the situation of those irrationals that were not equivalent to $\frac{\sqrt{5}-1}{2}$? Similarly, "Can the constant be improved for these irrationals?" (Prasad, 1948). Why my study intends to explore is also, to some extent, relate around the finding of this issue. However, the favorable response is a result of this theorem as mentioned below.

Theorem: 11 (Extension Theorem of Prasad, L.C. Eggan, 1961)

The aforesaid theorem was modified by L.C Eggan in 1961. "Let, suppose $\xi_2 = \sqrt{2} - 1$ and $\frac{a_i}{b_i}$ denote the j^{th} convergent to ξ_2. Then for any irrationals ξ which are not equivalent to $\frac{\sqrt{5}-1}{2}$ and possible integer *m*, are become relatively there are at least *m* solutions in relatively prime integers, *h*, *k* with *k > 0* to the inequality $\left|\xi - \frac{h}{k}\right| \leq \frac{1}{H_m k^2}$ ....(2)

Where $H_m = \sqrt{2} - 1 + \frac{a_{2m-1}}{b_{2m-1}}$,, Moreover if $\xi = \xi_2$ there are exactly *m* solutions" (Eggan, 1961). Here, Prasad's theorem with this result is distinct cases for the forthcoming theorem.

Theorem 12 L

L. C. Eggan in his theorem claims that if any ***n*** becomes non-negative integer, it becomes as $\xi_n = [0; n, n, n, n, \ldots] = \frac{\sqrt{n^2+4}-n}{2}$ **.** On the other hand, non-negative integer *m*, becomes as $H_m = \xi_n + n + \frac{a_{2m-1}}{b_{2m-1}}$, where $\frac{p_n}{q_n}$ are the j^{th} convergent of ξ_n. Likewise, if $\xi = [a_0; a_1, \ldots]$ is an irrational number with $a_j \geq n$ in the case of many *j* that are infinite have *m* solutions with comparatively "prime integers, *h*,*k with k > 0*,to the inequality $\left|\xi - \frac{h}{k}\right| \leq \frac{1}{H_m k^2}$ ". Additionally, "the constant H_m cannot be improved, since $\xi = \xi_n$ there are exactly *m* solutions" (Eggan, 1961) then equally achieved.

Corollary 6

If ξ is not equivalent to $\frac{\sqrt{5}-1}{2}$, minimum a pair of comparatively prime integer *h*, *k* with *k > 0* satisfying $\left|\xi - \frac{h}{k}\right| \leq \frac{1}{(\sqrt{2}+{}^{3}/_{2})k^2}$. Moreover $\xi = \sqrt{2} - 1$ there is precisely one pair.

Corollary 7

According to Perron**,** non-negative integer *n*, becomes $\xi = [0; n, n, n, \ldots]$. And, such prime integers as, *h*,*k* and *k > 0*, satisfying $\left|\xi - \frac{h}{k}\right| < \frac{1}{\left(\sqrt{n^2+4}\right)k^2}$ are determined (Perron, 1880-1975). However even Perron's Corollary has not confirmed about constant value of ξ.

Corollary 8

Regarding any non-negative "integer *n*, if ξ is as in theorem (12) but not equivalent to ξ_n, then there are infinitely many pairs of relatively prime integers" (Leggan,1961) *h, k* with *k > 0* satisfying $\left|\xi - \frac{h}{k}\right| < \frac{1}{\left(\frac{\sqrt{n^2+4}+n}{2}+\frac{1}{n}\right)k^2}$".

Theorem 13 [(Generalized Legendre's and Fatou's Theorem) Worley, 1981]

Bernadin Ibrahimpasic in his research article, "Explicit Version of Worley's Theorem n Diophantine Approximation" highlighting on Worley's theorem, focuses on ξ *as* "*a*real number" and "*h* and *k* as prime nonzero integers" being content with $\left|\xi - \frac{h}{k}\right| < \frac{A}{k^2}$," (Ibrahimpasic,2013). Here, *A* refers random non-negative original number. Worley's result in this line was somewhat enhanced by Dujella in 2004.

Theorem 14 [(Extension theorem of Worley) Dujella, 2004]:

Similarly, Dujella extended the theory of Worley more intensively**.** Let ξ "be a real number and let *h* and *k* be co-prime nonzero integers, satisfying $\left|\xi - \frac{h}{k}\right| < \frac{A}{k^2}$ (3)

Where A is a positive real number.

Then $(h,k)=(rp_{m+1} \pm sp_m, rp_{m+1} \pm sp_m)$, for some $m \geq -1$and non negative integers *r* and *s* such that $rs < 2A$" . (Cited in R.T. Worley, 1981, p 202-206).

Theorem 15: (Worley)

Whether ξ becomes "an irrational number", $A \geq$ "$\frac{1}{2}$ and $\frac{h}{k}$" is "a rational approximation to "ξ for which the inequality $\left|\xi - \frac{h}{k}\right| < \frac{A}{k^2}$" holds, then either" $\frac{h}{k}$ becomes convergent $\frac{p_m}{q_m}$ with ξ or $\frac{h}{k}$ contains the subsequent practices,

(i). $\frac{h}{k} = \frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}, r > s$ and $rs < 2A,$ and $r \leq s$ and $rs < A + \frac{r^2}{a_{m+2}}$.

(ii). $\frac{h}{k} = \frac{sp_{m+1}-tp_m}{sq_{m+1}-tq_m}, t > s$ and $ts < 2A,$ or $t \leq s$ and $ts\left(1-\frac{t}{2s}\right) < A$. Here, *r, s, t* are non-negative integers.

Because the fraction $\frac{h}{k}$ is in reduced form, it is clear that in the statements of theorems (2) and (4), we may assume that *gcd(r,s)=1* and *gcd(s,t)=1*. The corollary also gave the explicit version of his result for $k = 2$; $\left|\xi - \frac{h}{k}\right| < \frac{2}{k^2}$ implies

$\frac{h}{k} = \frac{p_m}{q_m}, \frac{p_{m+1}\pm p_m}{q_{m+1}\pm q_m}$

$\frac{2p_{m+1}\pm p_m}{2q_{m+1}\pm q_m}, \frac{3p_{m+1}+p_m}{3q_{m+1}+q_m}$

$\frac{p_{m+1}\pm 2p_m}{q_{m+1}\pm 2q_m}, \frac{p_{m+1}-3p_m}{q_{m+1}-3q_m}$ (Ibrahimpasic,2013. Cited from Worley, 1981, p202-206)

Theorem 16: (Explicit Version of Worley's Theorem)

Let $A \geq 3$ be a integer there exist a real number ξ and rational numbers $\frac{h_1}{k_1}$ and $\frac{h_2}{k_2}$ such that $\left|\xi - \frac{h_1}{k_1}\right| < \frac{A}{k_1^2}$, and $\left|\xi - \frac{h_2}{k_2}\right| < \frac{A}{k_2^2}$

Where $(h_1, k_1)=(rp_{m+1}+2p_m, rq_{m+1}+2p_m)$

and $(h_2,k_2)=(2p_{m+2} - tp_{m+1}, 2q_{m+2} - tp_{m+1})$

for some $m \geq -1$ and integers r and t such that $1 \leq r, t \leq A-1$ (Ibrahimpasic. B. 2013,p. 61).

Dujella and Ibrahimpasic gave the following result as mentioned in theorem 17.

Theorem 17

They jointly place the concept of real and rational number in an equality form. They say, "let A ∊ {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. If a real number ξ and a rational number, $\frac{h}{k}$ satisfy the inequality (1.2), then $\frac{h}{k} = \frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}$, where $(r,s) \in R_A = R_{A-1} \cup R_A^1$

Or $\frac{h}{k} = \frac{sp_{m+2}-tp_{m+1}}{sq_{m+2}-tq_{m+1}}$, where $(s,t) \in T_A = T_{A-1} \cup T_A^1$

(for an integer $(m \geq -1)$, where the sets R_A^1 and T_A^1 are given in the following table. Moreover, if any of the elements in sets R_A and T_A is omitted, the statement will no longer valid" (Ibrahimpasic. B. 2013,p. 61)

Table 1

A	R_A^1	T_A^1
3	{(1,3), (4,1), (5,1)}	{(3,1), (1,4), (1,5)
4	{(1,4), (3,2), (6,1), (7,1)}	{(4,1), (2,3), (1,6),(1,7)}
5	{(1,5), (2,3), (8,1), (9,1)}	{(5,1), (3,2), (1,8), (1,9)}
6	{(1,6), (5,2), (10,1), (11,1)}	{(6,1), (2,5), (1,10), (1,11)}
7	{(1,7), (2,5), (4,3), (12,1), (13,1)}	{(7,1), (5,2), (3,4), (1,12), (1,13)}
8	{(1,8),(3,4),(7,2),(14,1),(15,1)}	{(8,1),(4,3),(2,7),(1,14),(1,15)}
9	{(1,9),(5,3),(16,1),(17,1)}	{(9,1),(3,5),(1,16),(1,17)}
10	{(1,10),(9,2),(18,1),(19,1)}	{(10,1),(2,9),(1,18),(1,19)}
11	{(1,11),(2,7),(3,5),(20,1),(21,1)}	{(11,1),(7,2),(5,3),(1,20),(1,21)}
12	{(1,12),(5,4),(7,3),(11,2),(22,1),(23,1)}	{(12,1),(4,5),(3,7),(2,11),(1,22),(1,23)}

By theorem (2) and (4), what we reflect is on the pairs positive (r,s) and (s,t) being content with $rs < 2k, st < 2k, \gcd(r,s) = 1, \gcd(s,t) = 1$. In addition to it, in such case $k = 3$, this follows straightly with the "inequalities $r^2-sra_{m+2}+ka_{m+2} > 0$ (4)

And $a_{m+2} > \frac{t}{s}$ (5)

For $r=1$, respectively, $r=1$, that the pairs (r,s)=(1,s) and (s,t) = (s,1) with $s > k+1$ can be omitted similarly, for $r = 2$ or 3, resp. $t = 2$ or 3, we can exclude the pairs $(r,s) = (2,s)$ and $(s,t) = (s,2)$ with $s \geq \frac{A}{2} + 2$. and the pairs $(r,s) = (s,3)$ with $s \geq \frac{A}{3} + 3$." (Ibrahimpasic, 2013)

Theorem 18: [(Explicit Version of Worley's Theorem for k=13)

Bernadin Ibrahimpasic 's study concerns with real number and a rational number. He asserts that "a real number ξ and a rational number, $\frac{h}{k}$ satisfy the inequality" $\left|\xi - \frac{h}{k}\right| < \frac{13}{k^2}$, then $\frac{h}{k} = \frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}$,where $(r,s) \in R_{13} = R_{12} \cup \{(1,13),(3,7),(4,5),(24,1),(25,1)\}$,

Or $\frac{h}{k} = \frac{sp_{m+2}-tp_{m+1}}{sq_{m+2}-tq_{m+1}}$ where $(s,t) \in T_{13} = T_{12} \cup \{(13,1),(7,3),(5,4),(1,24),(1,25)\}$,

(for an integer $m \geq -1$). From the proof of the theorem (2) where *m* is the largest integer satisfying $\left|\xi - \frac{h}{k}\right| \leq \frac{p_m}{q_m}$. By theorem (2), it is to be thought that merely sets of positive (r, s) & (s, t) satiating "*rs<2k.st<2k,gcd(r, s)=1* and *gcd(s,t)=1* and *gcd(s,t)="1*. Four (4), the inequalities & t^2-sta_{m+2}+$ka_{m+2} > 0$ (6)

For r=1, respectively r=1, "use that these pairs (r,s)=(1,s) and (s,t)=(s,1) with s > k + 1 = 14 " gets omitted. Likewise, "for r=2 or 3, respectively t=2 or 3," it can be removed the "pairs (r,s) =(2,s) and (s,t) = (s,2) with s ≥ 13" /2+2. then "the pairs (r,s)=(3,s) and (s,t)=(s,3) with s ≥ 13/2+3. In particular, the pairs (r,s)=(2,9), (2,11), (3,8) and the pairs (s,t)=(9,2),(11,2),(8,3) can be excluded" Bernaidin Ibrahimpasic (2013). Here, what I intend to present is "(r,s)=(8,3) and (s,t)=(3,8)" (ibid) that becomes easy to replace with another pairs in minor results as rs, st respectively. On behalf of (r,s)=(8,3) & k=13 from (4) & (6), it can be got $\frac{8}{3} < \xi_{m+2} < \frac{64}{11}$, and therefore, I have three possibilities: If $\xi_{m+2} = 3$, then from (6),I get t = 3×3 -8 = 1, & I can substitute "(r,s) = (8,3) by (s,t)" = (3,1). When ($\xi_{m+2} = 4$, it can be changed with (s,t) = (3,4). Whether $\xi_{m+2} = 5$, I can replace with $(s, t) = 3{,}7)$. The evidence of such pairs (s,t)= (3,8) is totally similar. I use the inequalities (5) & (6) instead of" (ibid) $a_{m+2} > \frac{r}{s}$ and (4),we obtain $\frac{8}{3} < \xi_{m+2} < \frac{64}{11}$, and hence, again three possibilities are found as : "$\xi_{m+2} = 3, \xi_{m+2} = 4, \xi_{m+2} = 5$". In the same way, If $\xi_{m+2} = 3$, it can be got that t=1& it can replace it by (r,s)=(8,3) with (s, t)=(3,1). Similarly, $\xi_{m+2} = 4$ by replacing (s,t)= (3, 4). Again, in the condition $\xi_{m+2} = 5$, the possibility of (s,t)= (3,7)occurs (Bernaidin Ibrahimpasic (2013). The additional purpose of this research article is to present the pairs as (r, s) or (s, t) looking in the second intention, the account of the intention will be rather valid. To be more specific, when a pair can be excluded, $(r^1, s^1) \in R_{13}$, there happens the existence of the numbers like ξ & $\frac{h}{k}$ managing (3) properly. However $\frac{h}{k}$ becomes impossible to present in the form neither this $\frac{h}{k} = \frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}$, nor $\frac{h}{k} = \frac{sp_{m+2}-tp_{m+1}}{sq_{m+2}-tq_{m+1}}$ where for an integer $m > -1, (r, s) \in R_{13} - (r^1, s^1), (s, t) \in T_{13}$ and similarly for an excluded pair $(s^1, t^1) \in T_{13}$. The following table reflects the aforementioned possibilities.

Table 2

ξ	*h*	*k*	**M**	*r*	*S*	*t*
$\sqrt{5328}$	11533	158	1	1	13	12
$\sqrt{168}$	1063	82	1	3	7	4
$\sqrt{56}$	943	126	1	4	5	6
$\sqrt{626}$	30049	1201	0	24	1	26
$\sqrt{677}$	33851	1301	0	25	1	27
$\sqrt{5328}$	127957	1753	1	12	13	1
$\sqrt{168}$	1387	107	1	4	7	3
$\sqrt{56}$	1377	184	1	6	5	4
$\sqrt{626}$	32551	1301	0	26	1	24
$\sqrt{677}$	36557	1405	0	27	1	25
Source: (Bernadin Ibrahimpasic, 2013,p. 63)						

Here, If root 56 is converted into continued fraction, we get [7,2,14]. The same entity can be shown as $\sqrt{56}$ with their convergent, $\frac{7}{1}, \frac{15}{2}, \frac{217}{29}, \frac{449}{60}, \frac{6503}{869}, \ldots$ Its rational approximation $\frac{943}{126}$ as shown in the aforementioned table that can be " $\left|\sqrt{56} - \frac{943}{126}\right| \approx 0.0008123 < \frac{13}{126^2}$. "We have that the only representation of the fraction $\frac{943}{126}$ in the form $\frac{rp_{m+1}+sp_m}{rq_{m+1}+sq_m}, (r, s) \in$

$R_{13}\ Or\ \frac{sp_{m+2}-tp_{m+1}}{sq_{m+2}-tq_{m+1}}, (s,t) \in T_{13}\ is\ \frac{943}{126} = \frac{4.217+5.15}{4.29+5} = \frac{4.\sqrt{2}+5.\sqrt{1}}{4.k_2+5.k_1}$, which implies that the pair (4,5) cannot be excluded from the set R_{13}. (Ibrahimpasic, 2013)

5. CONCLUSIONS

The study of Hurwitz's Theorem has been diversified in the field of number theory issuing more on "approximation of irrational numbers by rational numbers" since its first inception by Greek Mathematician Diophantus in 250 A.D. The study has met these primary concerns − The Irrationals and Dirichlet's, Extension of Hurwitz Theorem and its connectivity with Continued Fractions− so as to envision its theoretical findings and its implementation. The study has assumed to be both essential and applicable in the practical use of irrational numbers concern to the area of number theory and other similar fields. What I have analyzed through the study is theorizing the way Hurwitz has highlighted as: "Let ξ be an irrational number, with its categories as: (i) There are infinitely many rational numbers $\frac{h}{k}$ such that: $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$ and (ii) If $\sqrt{5}$ is replaced by $C > \sqrt{5}$, then there are irrational numbers ξ for which statement (i) does not hold" (Ivan N, 10). Accordingly, the study has found out the paradigm of expansion and ongoing modification of the aforementioned "Hurwitzian Theorem" has comprised multi-area of mathematics. The expected output of the study is related to carry out research in the field of mathematics in Nepal so as to up bring the research quality adjoining to international level.

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